

THE MUTUAL INCLUSION IN A NONLOCAL COMPETITIVE LOTKA VOLTERRA SYSTEM

XIAOJIE HOU¹, BIAO WANG², ZHENGCE ZHANG^{2,*}

ABSTRACT. We investigate the traveling front solutions of a nonlocal Lotka Volterra system to illustrate the outcome of the competition between two species. The existence of the front solution is obtained through a new monotone iteration scheme, the uniqueness of the front solution corresponding to each propagation speed is proved by sliding domain method adapted to nonlocal systems, and the asymptotic decay rate of the fronts with critical and noncritical wave speeds is derived by a new method, which is different from the single equation case. The results demonstrate that in the long run, two weakly competing species can co-exist.

1. INTRODUCTION

In this paper we study the properties of traveling front solutions of the following nonlocal Lotka-Volterra competition system:

$$(1.1) \quad \begin{cases} \hat{u}_t = J * \hat{u} - \hat{u} + \hat{u}(1 - \hat{u} - a_1 \hat{v}), \\ \hat{v}_t = J * \hat{v} - \hat{v} + r\hat{v}(1 - a_2 \hat{u} - \hat{v}), \end{cases} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+$$

where $'*$ ' denotes convolution: $J * w = \int_{\mathbb{R}} J(x-y)w(t, y)dy$. The nonnegative functions $\hat{u}(x, t)$ and $\hat{v}(x, t)$ are population densities of the two competing species. The constant $r > 0$ is the relative growth rate of species \hat{v} , and $a_1, a_2 > 0$ are interactive constants. The integration kernel J is a probability density function satisfying $\int_{\mathbb{R}} J = 1$, being radial and nonnegative with finite support of nonzero measure. Depending on the conditions of a_1 and a_2 , system (1.1) may have four constant equilibrium states $O(0, 0)$, $A(1, 0)$, $B(0, 1)$ and $C(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2})$. We can then classify their stability as follows:

- (i) If $0 < a_1, a_2 < 1$, (1.1) has four equilibria with $(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2})$ being asymptotically stable, and $(0, 0)$, $(1, 0)$, $(0, 1)$ being unstable;
- (ii) If $0 < a_1 < 1 < a_2$, (1.1) has three equilibria: $(1, 0)$ is asymptotically stable, while $(0, 1)$ and $(0, 0)$ are unstable;
- (iii) If $a_1, a_2 > 1$, then $(1, 0)$ and $(0, 1)$ are both stable and $(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2})$, $(0, 0)$ are unstable;

2000 *Mathematics Subject Classification.* Primary 35B35, Secondary 91B18, 35K57, 35B40.

Key words and phrases. Nonlocal Lotka Volterra, Traveling front, Asymptotic behavior, Uniqueness.

(iv) If $0 < a_2 < 1 < a_1$, then $(0, 1)$ is asymptotically stable, while $(0, 0)$ and $(1, 0)$ are unstable.

We are interested in the case (i) so the condition $0 < a_1, a_2 < 1$ is assumed throughout the paper. The cases (ii) and (iv) are currently being investigated in [16].

System (1.1) describes the long range competition between two species, so a natural question is who will be the winner in the long run. We will address this question, and others such as the uniqueness, the monotonicity as well as the asymptotics by studying the traveling front solutions of (1.1). The traveling front solution of (1.1) is a pair of nonnegative smooth functions $(u(x, t), v(x, t)) = (u(x + ct), v(x + ct)) := (u(\xi), v(\xi))$, which satisfies the following boundary value problem

$$(1.2) \quad \begin{cases} J * \hat{u} - \hat{u} - c\hat{u}' + \hat{u}(1 - \hat{u} - a_1\hat{v}) = 0, \\ J * \hat{v} - \hat{v} - c\hat{v}' + r\hat{v}(1 - a_2\hat{u} - \hat{v}) = 0, \\ (\hat{u}, \hat{v})(-\infty) = (1, 0), (\hat{u}, \hat{v})(+\infty) = (\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}) := (k_1, k_2) \end{cases}$$

where $\xi = x + ct$, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$ is the moving coordinate and $c > 0$ is the speed of the front.

The nonlocal reaction diffusion equations and systems can be found in many applications such as cell biology, phase transition, ecology, and the neurons and neuronal network [1, 2, 5, 11, 20, 13, 25, 21, 8, 23, 7]. One of the important features of those models is the appearance of traveling front solutions, which is frequently used to study the transition between two equilibrium states, see ([14]). In recent years, there have been fruitful results concerning traveling front solutions of the nonlocal equations and systems. For example in [11] the authors systematically investigated the traveling front solutions of integro-differential equation

$$(1.3) \quad u_t(x, t) = J * u(x, t) - u(x, t) + f(u(x, t)), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+$$

and established the existence and uniqueness of the traveling front solution as well as the asymptotic behaviors for the ignition type, the bistable type and the KPP (Kolmogorov-Petrovsky-Piskounov) type nonlinearities. And in [9], the nonlocal diffusion-reaction equations of the form

$$(1.4) \quad \begin{cases} J * u(x) - u(x) - cu'(x) + f(u(x)) = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \mathbb{R} \setminus \Omega \end{cases}$$

where Ω is a bounded interval, was considered. In particular, the maximum principle as well as the sliding domain method were derived, and successfully applied to obtain the uniqueness and monotone behavior of positive solution of (1.4). Recently, Pan and Li et al [22] established the existence of the traveling wave fronts in nonlocal delayed reaction-diffusion systems by using the upper-lower solution method and Schauder's fixed point theorem. G. Lv [19] dealt with the asymptotic behavior of the traveling fronts and entire solutions for a nonlocal monostable equation. By using lower and upper traveling wave solutions, the authors in [25] proved the existence of traveling wave

solutions for the two-species integro-differential Lotka-Volterra competition model in the form similar to (1.1). For the recent work concerning the non-local reaction-diffusion equations and systems, refer to [25, ?, 21, 22, 7, 24] and the references therein.

We note that other methods, such as the homotopy continuation method [8, 2], and the contraction mapping method [20], could also be used to establish the existence of traveling front solutions for nonlocal equations and systems. Here we would like to propose a new and easy monotone iteration scheme to derive the existence of traveling wave solutions of (1.1). To this end, we first change (1.2) into a local monotone type by setting $u = 1 - \hat{u}$, $v = \hat{v}$, so this leads to

$$(1.5) \quad \begin{cases} J * u - u - cu' + (u - 1)(u - a_1 v) = 0, \\ J * v - v - cv' + rv(1 - a_2 + a_2 u - v) = 0, \\ (u, v)(-\infty) = (0, 0), \quad (u, v)(+\infty) = (a_1 k_2, k_2). \end{cases}$$

Observing that if u is replaced by $a_1 v$ in the second equation of (1.5), we recover the nonlocal KPP equation. And the existence, uniqueness as well as the asymptotic behaviors of the traveling front solutions are known recently ([9, 19, 4]). The construction of the upper and lower solutions for (1.5) is based on this simple observation.

The asymptotic behavior closely ties to the other properties of the front solutions such as their uniqueness and the stability. The asymptotics of the traveling wave solutions is investigated by several steps of integral inequalities. Our method differs from that in [4, 19], where Ikehara's Tauberian Theorem can be applied directly. In the current system the method fails due to the nontrivial coupling of the two components. We solve this problem by first studying the canonical form of the linearized system, then switching back to the original system to have the desired results.

For the proof of the uniqueness of traveling front solutions of (1.5), we adapt the generalized sliding domain method [16]. Combining the sliding domain method and the nonlocal comparison principle we are able to establish the uniqueness (up to a shift of the origin) of the traveling wave solutions for each front speed.

The outline of the paper is as follows. In Section 2 we set up the upper and lower solutions and then apply monotone iteration scheme to derive the existence of the front solutions. In Section 3 we prove the strict monotonicity of the traveling front solutions by comparison principle and obtain the asymptotic decay rates, and derive the uniqueness of the front solutions by a generalized sliding domain method.

2. EXISTENCE OF FRONT SOLUTIONS

In this section, we first introduce some lemmas to aid set up the upper and lower solution pairs for (1.5). The existence of traveling front solutions is then established by monotone iteration ([22]). For the rest of the paper the inequality between two vectors is component-wise.

Definition 1. A smooth function $(\bar{u}(\xi), \bar{v}(\xi))^T$, $\xi \in \mathbb{R}$ is an upper solutions of (1.5) if it satisfies

$$(2.1) \quad \begin{cases} J * u - u - cu' + (u - 1)(u - a_1 v) \leq 0, \\ J * v - v - cv' + rv(1 - a_2 + a_2 u - v) \leq 0 \end{cases}$$

and the boundary conditions

$$(2.2) \quad (u, v)(-\infty) \geq (0, 0), \quad (u, v)(+\infty) \geq (a_1 k_2, k_2).$$

A lower solution of (1.5) is defined in a similar way by reversing the inequalities in (2.1) and (2.2).

The construction of upper and lower solutions pairs is based on the solutions of the nonlocal KPP equation of the form:

$$(2.3) \quad \begin{cases} J * u - u - cu' + f(u) = 0, \\ u(-\infty) = 0, \quad u(+\infty) = b > 0, \end{cases}$$

where the smooth function f satisfies $f(0) = f(b) = 0$, $f(s) > 0$ for $s \in (0, b)$, $f'(0) > 0$ and f is non-increasing near b , the integral kernel J satisfies the conditions specified in section 1.

Let the constant c^* be defined as

$$c^* =: \min \left\{ \lambda > 0 : \frac{1}{\lambda} \left[\int_{\mathbb{R}} J(\xi) e^{\lambda \xi} dx + f'(0) - 1 \right] \right\},$$

then $c^* > 0$ and is finite ([23]). We recall the following result:

Lemma 2. *Let f be defined as above, then for any $c \geq c^*$ system (2.3) has a unique (up to a shift of origin) smooth and monotonically increasing solution $w(\xi)$, $\xi \in \mathbb{R}$.*

Proof. See [23, Theorem 2.1]. □

To construct the upper solution for the system (1.5), we begin with the following version of nonlocal KPP equation:

$$(2.4) \quad \begin{cases} J * \tilde{v} - \tilde{v} - c\tilde{v}' + r(1 - a_2)\tilde{v}(1 - \frac{\tilde{v}}{k_2}) = 0, \\ \tilde{v}(-\infty) = 0, \quad \tilde{v}(+\infty) = k_2 \end{cases}$$

where corresponding to (2.3) $f(\tilde{v}) = r\tilde{v}(1 - a_2 + a_1 a_2 \tilde{v} - \tilde{v}) > 0$ for $\tilde{v} \in (0, k_2)$, recalling that $k_2 = \frac{1 - a_2}{1 - a_1 a_2}$. It is easy to see that $f(0) = f(k_2) = 0$, $f'(0) = r(1 - a_2) > 0$ and $f'(k_2) = -r(1 - a_2) < 0$. According to Lemma 2, for each fixed $c \geq c^*$ with

$$(2.5) \quad c^* =: \min \left\{ \lambda > 0 : \frac{1}{\lambda} \left[\int_{\mathbb{R}} J(\xi) e^{\lambda \xi} dx + r(1 - a_2) - 1 \right] \right\} > 0.$$

system (2.4) has a unique (up to a translation of the origin) traveling front solution \bar{v} satisfying the given boundary conditions.

Define

$$(2.6) \quad \begin{pmatrix} \bar{u}(\xi) \\ \bar{v}(\xi) \end{pmatrix} = \begin{pmatrix} a_1 \bar{v}(\xi) \\ \bar{v}(\xi) \end{pmatrix}, \quad \xi \in \mathbb{R},$$

then we have the following result,

Lemma 3. *Let $c^* > 0$ be defined as in (2.5) then for each fixed $c \geq c^*$, (2.6) is a smooth upper solution for system (1.5).*

Proof. On the boundary one has

$$\begin{pmatrix} \bar{u}(-\infty) \\ \bar{v}(-\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \bar{u}(+\infty) \\ \bar{v}(+\infty) \end{pmatrix} = \begin{pmatrix} a_1 k_2 \\ k_2 \end{pmatrix}.$$

As for the u component, we have

$$\begin{aligned} & J * \bar{u} - \bar{u} - c\bar{u}' + (\bar{u} - 1)(\bar{u} - a_1 \bar{v}) \\ &= a_1(J * \bar{v} - \bar{v} - c\bar{v}') + (a_1 \bar{v} - 1)(a_1 \bar{v} - a_1 \bar{v}) \\ &= -a_1 r \bar{v}(1 - a_2 + a_1 a_2 \bar{v} - \bar{v}) \\ &= -a_1 r(1 - a_2) \bar{v}(k_2 - \bar{v}) \leq 0 \end{aligned}$$

We then verify the second component in (1.5),

$$\begin{aligned} & J * \bar{v} - \bar{v} - c\bar{v}' + r\bar{v}(1 - a_2 + a_1 a_2 \bar{v} - \bar{v}) \\ &= -r\bar{v}(1 - a_2 + a_1 a_2 \bar{v} - \bar{v}) + r\bar{v}(1 - a_2 + a_1 a_2 \bar{v} - \bar{v}) \\ &= 0 \end{aligned}$$

Thus (\bar{u}, \bar{v}) forms a smooth upper-solution for (1.5). □

We next construct the lower solution pair for system (1.5). We choose a number l according to one of the following conditions:

$$(2.7) \quad \begin{cases} 1). 0 < l < \max\{\frac{a_1 - r}{a_1(1 - ra_2)}, \frac{1}{1 + r(1 - a_2) + k_2(a_1 - r)}\} \text{ if } 0 < r < a_1. \\ 2). 0 < l < \frac{1}{1 + r(1 - a_2)} \text{ if } a_1 \leq r \leq \frac{1}{a_2}; \\ 3). 0 < l < \max\{\frac{r - a_1}{a_1(ra_2 - 1)}, \frac{1}{1 + r(1 - a_2)}\} \text{ if } r > \frac{1}{a_2}, \end{cases}$$

and work with another nonlocal KPP equation:

$$(2.8) \quad \begin{cases} J * \tilde{v} - \tilde{v} - c\tilde{v}' + r(1 - a_2)\tilde{v}(1 - \frac{1 - a_1 a_2 l}{1 - a_2} \tilde{v}) = 0, \\ \tilde{v}(-\infty) = 0, \tilde{v}(+\infty) = \frac{1 - a_2}{1 - a_1 a_2 l} < k_2. \end{cases}$$

Corresponding to the notions in Lemma 2,

$$f(\tilde{v}) = r(1 - a_2)\tilde{v}\left(1 - \frac{1 - a_1 a_2 l}{1 - a_2} \tilde{v}\right) > 0$$

for $\tilde{v} \in (0, \frac{1 - a_2}{1 - a_1 a_2 l})$, and $f(0) = f(\frac{1 - a_2}{1 - a_1 a_2 l}) = 0$, $f'(0) = r(1 - a_2) > 0$ and $f'(\frac{1 - a_2}{1 - a_1 a_2 l}) = -r(1 - a_2) < 0$. Noting we have the same c^* as in the previous Lemma.

For each fixed $c \geq c^*$, define

$$(2.9) \quad \begin{pmatrix} \underline{u}(\xi) \\ \underline{v}(\xi) \end{pmatrix} = \begin{pmatrix} a_1 l \underline{v}(\xi) \\ \underline{v}(\xi) \end{pmatrix}, \quad \xi \in \mathbb{R},$$

where $\underline{v}(\xi)$ is solution of (2.8).

Lemma 4. *For each fixed $c \geq c^*$, (2.9) is a smooth lower solution of system (1.5).*

Proof. On the boundary one has

$$\begin{pmatrix} \overline{u}(-\infty) \\ \overline{v}(-\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \overline{u}(+\infty) \\ \overline{v}(+\infty) \end{pmatrix} = \begin{pmatrix} a_1 \frac{1-a_2}{1-a_1 a_2 l} \\ \frac{1-a_2}{1-a_1 a_2 l} \end{pmatrix} \leq \begin{pmatrix} a_1 k_2 \\ k_2 \end{pmatrix},$$

due to the fact $\frac{1-a_2}{1-a_1 a_2 l} < k_2$.

Furthermore,

$$\begin{aligned} & J * \underline{v} - \underline{v} - c \underline{v}' + r \underline{v}(1 - a_2 + a_2 \underline{u} - \underline{v}) \\ &= -r(1 - a_2) \underline{v} \left(1 - \frac{1 - a_1 a_2 l}{1 - a_2} \underline{v} \right) + r \underline{v}(1 - a_2 + a_1 a_2 l \underline{v} - \underline{v}) \\ &= 0, \end{aligned}$$

Next, we check lower solution with respect to u , the computation as follows:

$$\begin{aligned} & J * \underline{u} - \underline{u} - c \underline{u}' + (\underline{u} - 1)(\underline{u} - a_1 \underline{v}) \\ &= a_1 l (J * \underline{v} - \underline{v} - c \underline{v}') + (a_1 l \underline{v} - 1)(a_1 l \underline{v} - a_1 \underline{v}) \\ &= a_1 l \left[-r(1 - a_2) \underline{v} \left(1 - \frac{1 - a_1 a_2 l}{1 - a_2} \underline{v} \right) \right] + a_1 \underline{v} (a_1 l \underline{v} - 1)(l - 1) \\ &= a_1 \underline{v} \{ -rl(1 - a_2) - l + 1 + [-rl(a_1 a_2 l - 1) + a_1 l(l - 1)] \underline{v} \} \\ &\geq 0 \end{aligned}$$

by the condition (2.7). Hence, the conclusion of the lemma follows. \square

We next show that the upper and lower solutions as obtained in Lemmas 3 and 4 are ordered. This can be achieved by shifting the upper solution to the left along the axis, but first a comparison of the asymptotic rates of the upper and lower solutions at infinities is needed. Consider the following characteristic equations at $\pm\infty$:

$$M_-(\lambda) = \int_{\mathbb{R}} J(\xi) e^{\lambda \xi} d\xi - c\lambda + f'(0) - 1, \quad \text{at } -\infty,$$

and

$$M_+(\lambda) = \int_{\mathbb{R}} J(\xi) e^{\lambda \xi} d\xi - c\lambda + f'(b) - 1 \quad \text{at } +\infty.$$

It is easy to verify that $M_-(\lambda)$ and $M_+(\lambda)$ ([23] and [4]) have the following properties:

(a) The characteristic equation $M_-(\lambda)$ for nonlocal reaction-diffusion equations (2.3) (at $-\infty$) has two positive roots $\lambda_1(c) < \lambda_2(c)$ for $c > c^*$ and one double root $\lambda^*(c) > 0$ for $c = c^*$ and no real root for $0 < c < c^*$;

(b) The characteristic equation $M_+(\lambda)$ (at $+\infty$) has one negative root $\lambda_3(c)$ and one positive root $\lambda_4(c)$ for any $c > 0$;

The next lemma specifies the asymptotic behaviors of the front solutions of the KPP equations ([19] and [4]):

Lemma 5. *The traveling front solution $w(\xi)$ as in Lemma 2 has the following asymptotic behaviors:*

For the critical front with speed $c = c^$,*

$$w(\xi) = b_w \xi e^{\lambda^*(c)\xi} + o(\xi e^{\lambda^*(c)\xi}), \quad \xi \rightarrow -\infty$$

$$w(\xi) = b - d_w e^{\lambda_3(c^*)\xi} + o(e^{\lambda_3(c^*)\xi}), \quad \xi \rightarrow +\infty.$$

and for the noncritical front with $c > c^$,*

$$w(\xi) = a_w e^{\lambda_1(c)\xi} + o(e^{\lambda_1(c)\xi}), \quad \xi \rightarrow -\infty;$$

$$w(\xi) = b - c_w e^{\lambda_3(c)\xi} + o(e^{\lambda_3(c)\xi}), \quad \xi \rightarrow +\infty;$$

where a_w, c_w, d_w are positive constants and b_w is negative.

By shifting the upper solution far enough to the left, the upper- and lower-solutions as derived in Lemma 3 and Lemma 4 are ordered.

Lemma 6. *Let $c \geq c^*$ be fixed and $(\bar{u}, \bar{v})^T, (\underline{u}, \underline{v})^T$ be the upper and lower solutions defined in (2.6) and (2.9), then there exists a number $\tau \geq 0$ such that*

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(\xi + \tau) \geq \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}(\xi) \quad \text{for } \xi \in \mathbb{R}.$$

Proof. We only show the lemma for $c > c^*$ since the one for $c = c^*$ is similar. By Lemma 5,

$$(2.10) \quad \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(\xi) = \begin{pmatrix} a_1 A_1 \\ A_1 \end{pmatrix} e^{\lambda_1 \xi} + o(e^{\lambda_1 \xi})$$

and

$$(2.11) \quad \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}(\xi) = \begin{pmatrix} a_1 l B_1 \\ B_1 \end{pmatrix} e^{\lambda_1 \xi} + o(e^{\lambda_1 \xi})$$

as $\xi \rightarrow -\infty$; and

$$(2.12) \quad \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(\xi) = \begin{pmatrix} a_1 k_2 \\ k_2 \end{pmatrix} - \begin{pmatrix} a_1 k_2 A_2 \\ k_2 A_2 \end{pmatrix} e^{\lambda_3 \xi} + o(e^{\lambda_3 \xi})$$

and

$$(2.13) \quad \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}(\xi) = \begin{pmatrix} a_1 l \\ l \end{pmatrix} \frac{1 - a_2}{1 - a_1 a_2 l} - \begin{pmatrix} a_1 l B_1 \\ B_2 \end{pmatrix} e^{\lambda_3 \xi} + o(e^{\lambda_3 \xi})$$

as $\xi \rightarrow +\infty$, where $\lambda_1 > 0$ is the smaller positive root of the characteristic equation

$$M_-(\lambda) = \int_{\mathbb{R}} J(\xi) e^{\lambda \xi} d\xi - c\lambda + r(1 - a_2) - 1,$$

and $\lambda_3 < 0$ is the negative root of the characteristic equation

$$M_+(\lambda) = \int_{\mathbb{R}} J(\xi) e^{\lambda \xi} d\xi - c\lambda - r(1 - a_2) - 1,$$

and A_1, A_2, B_1, B_2 are positive constants.

Since (2.4) is translation invariant, $\bar{v}^{\tilde{\tau}}(\xi) =: \bar{v}(\xi + \tilde{\tau})$ is also a solution of (2.4) for any $\tilde{\tau} \in \mathbb{R}$. It then follows that $(\bar{u}^{\tilde{\tau}}, \bar{v}^{\tilde{\tau}})^T(\xi)$ is an upper solution for system (1.5). For the asymptotic behavior of $(\bar{u}^{\tilde{\tau}}, \bar{v}^{\tilde{\tau}})^T(\xi)$ at $-\infty$, we can simply replace $(a_1 A_1, A_1)^T$ by $(a_1 A_1, A_1)^T e^{\lambda_1 \tilde{\tau}}$ in (2.10). Now we choose $\tilde{\tau} > 0$ large enough such that

$$\begin{pmatrix} a_1 A_1 \\ A_1 \end{pmatrix} e^{\lambda_1 \tilde{\tau}} > \begin{pmatrix} a_1 l B_1 \\ B_1 \end{pmatrix}.$$

Then there exists a sufficiently large $N_1 > 0$ such that

$$(2.14) \quad \begin{pmatrix} \bar{u}^{\tilde{\tau}}(\xi) \\ \bar{v}^{\tilde{\tau}}(\xi) \end{pmatrix} > \begin{pmatrix} \underline{u}(\xi) \\ \underline{v}(\xi) \end{pmatrix} \text{ for } \xi \in (-\infty, -N_1].$$

On the other hand, the boundary conditions of upper and lower solutions at $+\infty$ also imply that there exists a number $N_2 > 0$ such that

$$(2.15) \quad \begin{pmatrix} \bar{u}^{\tilde{\tau}}(\xi) \\ \bar{v}^{\tilde{\tau}}(\xi) \end{pmatrix} > \begin{pmatrix} \underline{u}(\xi) \\ \underline{v}(\xi) \end{pmatrix} \text{ for } \xi \in [N_2, +\infty).$$

We next show that the inequalities (2.14) and (2.15) also hold on the interval $[-N_1, N_2]$. There are two possible cases to deal with:

Case 1. If we already have

$$(2.16) \quad \begin{pmatrix} \bar{u}^{\tilde{\tau}}(\xi) \\ \bar{v}^{\tilde{\tau}}(\xi) \end{pmatrix} > \begin{pmatrix} \underline{u}(\xi) \\ \underline{v}(\xi) \end{pmatrix} \quad \text{on } [-N_1, N_2].$$

we then let $\tau = \tilde{\tau}$ and have the conclusion.

Case 2. There exists a point $\xi_0 \in (-N_1, N_2)$ such that

$$(2.17) \quad \begin{pmatrix} \bar{u}^{\tilde{\tau}}(\xi_0) \\ \bar{v}^{\tilde{\tau}}(\xi_0) \end{pmatrix} \leq \begin{pmatrix} \underline{u}(\xi_0) \\ \underline{v}(\xi_0) \end{pmatrix}$$

with strict inequality holding for at least one of the two components.

In this case, we can use the sliding domain method [16]. We first shift $(\bar{u}^{\tilde{\tau}}, \bar{v}^{\tilde{\tau}})^T(\xi)$ to the left by increasing $\tilde{\tau}$ until we can find a $\tau_1 > \tilde{\tau} > 0$ such that $(\bar{u}^{\tau_1}(\xi), \bar{v}^{\tau_1}(\xi))^T > (\underline{u}(\xi), \underline{v}(\xi))^T$ on the interval $[-N_1, N_2 - (\tau_1 - \tilde{\tau})]$. we then shift $(\bar{u}^{\tau_1}(\xi), \bar{v}^{\tau_1}(\xi))^T$ back to the right by increasing τ_1 to some $\tau_2 > \tilde{\tau}$ such that one of the branches of the upper solution tangents to its counterpart of the lower solution at some point ξ_2 in the interval $(-N_1 + \tau_2, N_2 - (\tau_1 - \tilde{\tau}))$. On the endpoints of the interval $(-N_1 + \tau_2, N_2 - (\tau_1 - \tilde{\tau}))$, we still have $(\bar{u}^{\tau_2}(\xi), \bar{v}^{\tau_2}(\xi))^T > (\underline{u}(\xi), \underline{v}(\xi))^T$. In summary, we now have $\bar{u}^{\tau_2}(\xi_2) = \underline{u}(\xi_2)$ and $\bar{u}^{\tau_2}(\xi) \geq \underline{u}(\xi), \bar{v}^{\tau_2}(\xi) \geq \underline{v}(\xi)$ for $\xi \in (-N_1 + \tau_2, N_2 - (\tau_1 - \tilde{\tau}))$.

Let $W(\xi) := (\bar{u}^{\tau_2}, \bar{v}^{\tau_2})^T(\xi) - (\underline{u}, \underline{v})^T(\xi)$ and $F = (F_1, F_2)^T = ((u-1)(u-a_1v), rv(1-a_2+a_2u-v))^T$. For $\xi \in (-N_1 + \tau_2, N_2 - (\tau_1 - \tilde{\tau}))$ we have

$$(2.18) \quad \begin{aligned} J * w_1 - w_1 - cw'_1 + \frac{\partial F_1}{\partial u}(\underline{u} + \zeta_1 w_1, \bar{v})w_1 + \frac{\partial F_1}{\partial v}(\bar{u}, \underline{v} + \zeta_2 w_2)w_2 &\leq 0, \\ J * w_2 - w_2 - cw'_2 + \frac{\partial F_2}{\partial u}(\underline{u} + \zeta_3 w_1, \bar{v})w_1 + \frac{\partial F_2}{\partial v}(\bar{u}, \underline{v} + \zeta_4 w_2)w_2 &\leq 0 \end{aligned}$$

for some $\zeta_i \in [0, 1]$, $i = 1, 2, 3, 4$. Since the above system is monotone we further have

$$(2.19) \quad J * w_1 - w_1 - cw'_1 + \frac{\partial F_1}{\partial u}(\underline{u} + \zeta_1 w_1, \bar{v})w_1 \leq 0.$$

That ξ_2 is a global minimum point for $w_1(\xi)$ implies $w'_1(\xi_2) = 0$. Hence at $\xi = \xi_2$ we have

$$(2.20) \quad J * w_1(\xi_2) - w_1(\xi_2) - cw'_1(\xi_2) + \frac{\partial F_1}{\partial u}(\underline{u} + \zeta_1 w_1, \bar{v})w_1(\xi_2) = J * w_1(\xi_2) > 0.$$

This contradiction shows that such ξ_2 does not exist, and therefore we can further decrease τ_2 to $\tilde{\tau}$. This shows that ξ_0 does not exist either, we therefore have

$$\begin{pmatrix} \bar{u}^{\tilde{\tau}}(\xi) \\ \bar{v}^{\tilde{\tau}}(\xi) \end{pmatrix} \geq \begin{pmatrix} \underline{u}(\xi) \\ \underline{v}(\xi) \end{pmatrix} \quad \text{for } \xi \in \mathbb{R}.$$

□

We still use $(\bar{u}, \bar{v})^T$ to denote the shifted upper solution as given in Lemma 3. With the ordered upper and lower solutions in Lemma 6 and the monotone iteration scheme [22], we have the following result on existence of the traveling wave solution of (1.5).

Theorem 7. *Assume that $0 < a_1, a_2 < 1$ and (2.7), for each $c \geq c^*$ with c^* being defined in (2.5), system 1.5 exists a traveling wave solution $(u(x, t), v(x, t)) = (u(x + ct), v(x + ct))$ connecting $(0, 0)$ and $(a_1 k_2, k_2)$.*

Proof. Applying the monotone iteration method [22] of the ordered upper and lower solution pairs obtained in Lemma 3 and Lemma 4, we then conclude the Theorem. □

Remark 8. As pointed out in the introduction, there are several methods to show the existence of the traveling wave solutions, and the above proposed construction of the upper and lower solution is one of the easiest; and furthermore, the method can also provide some insight into the asymptotic behavior of the traveling front solutions, as it will be shown in the next section.

3. ASYMPTOTIC BEHAVIOR AND UNIQUENESS OF THE FRONT SOLUTIONS

The monotonicity of the front solutions is a direct consequence of the nonlocal comparison principle and the monotone iteration.

Theorem 9. *The traveling front solution of (1.5) derived in Theorem 7 is strictly monotonically increasing from $-\infty$ to $+\infty$.*

Proof. By the monotone iteration process [22], the traveling wave solution $U(\xi) = (u(\xi), v(\xi))^T$ is increasing for $\xi \in \mathbb{R}$, then its derivative satisfies $(g_1(\xi), g_2(\xi))^T = U'(\xi) \geq 0$, and

$$(3.1) \quad J * g_1 - g_1 - cg_1' + \frac{\partial F_1}{\partial u}(u, v)g_1 + \frac{\partial F_1}{\partial v}(u, v)g_2 = 0,$$

$$(3.2) \quad J * g_2 - g_2 - cg_2' + \frac{\partial F_2}{\partial u}(u, v)g_1 + \frac{\partial F_2}{\partial v}(u, v)g_2 = 0,$$

and

$$(3.3) \quad (g_1(\xi), g_2(\xi))^T \geq 0, \quad (g_1, g_2)^T(\pm\infty) = 0.$$

The local monotone structure of (3.1) and (3.2) as well as the maximum principle ([10]) implies that $(g_1, g_2)^T(\xi) > 0$ for $\xi \in \mathbb{R}$. This concludes the strict monotonicity of the traveling wave solutions. \square

We next derive the asymptotic behaviors of traveling front solution at $\pm\infty$. Such result for the single nonlocal equation with KPP or bistable nonlinearity is already known ([4, 19]), but for the system with nontrivial coupling such as (1.1) it is new.

Since the asymptotic behavior at $-\infty$ can be obtained by a straightforward comparison, we only concentrate on deriving the asymptotic behavior at $+\infty$.

To get the asymptotic decay rate of the traveling wave solutions at $\xi \rightarrow +\infty$, we set $c \geq c^*$ and use the vector form to derive the results. Firstly, changing the system (1.5) by the transformations:

$$\tilde{u} = a_1 k_2 - u, \quad \tilde{v} = k_2 - v$$

and reversing the sign of ξ , we will have the following system

$$(3.4) \quad \begin{cases} -J * \tilde{u} + \tilde{u} - c\tilde{u}' + (-k_1 - \tilde{u})(a_1 \tilde{v} - \tilde{u}) = 0, \\ -J * \tilde{v} + \tilde{v} - c\tilde{v}' + r(k_2 - \tilde{v})(-a_2 \tilde{u} + \tilde{v}) = 0 \end{cases}$$

and yet the boundary conditions are still

$$(3.5) \quad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(+\infty) = \begin{pmatrix} a_1 k_2 \\ k_2 \end{pmatrix}.$$

We only need to study the asymptotic behavior of the system (3.4) and (3.5) at $\xi \rightarrow -\infty$. Once this is done, on switching back to the original variables we have the asymptotic behavior at $+\infty$.

Lemma 10. *Let $c \geq c^*$ be fixed then there exists a number $\gamma > 0$ such that the solutions of (3.4) and (3.5) satisfy*

$$\begin{pmatrix} \tilde{u}(\xi) \\ \tilde{v}(\xi) \end{pmatrix} \leq O(e^{\gamma\xi}) \quad \text{as } \xi \rightarrow -\infty.$$

Remark 11. This Lemma says that the traveling front solution of system (1.5) approach the equilibrium $(a_1 k_2, k_2)$ at least exponentially fast for $\xi \rightarrow +\infty$.

Proof. We divide the proof into three steps:

Step 1. $(\tilde{u}(\xi), \tilde{v}(\xi))$ is integrable for ξ close to $-\infty$.

For ξ negatively large the system (3.4) satisfies

$$(3.6) \quad \begin{aligned} c \begin{pmatrix} \tilde{u}' \\ \tilde{v}' \end{pmatrix} &= -J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} (-k_1 - \tilde{u})(a_1 \tilde{v} - \tilde{u}) \\ r(k_2 - \tilde{v})(-a_2 \tilde{u} + \tilde{v}) \end{pmatrix} \\ &\geq -J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \end{aligned}$$

The above inequality is true because the limit of the Jacobian of the vector $((-k_1 - \tilde{u})(a_1 \tilde{v} - \tilde{u}), r(k_2 - \tilde{v})(-a_2 \tilde{u} + \tilde{v}))$ as $\xi \rightarrow -\infty$ satisfies componentwisely the inequality

$$\begin{pmatrix} k_1 & -a_1 k_1 \\ -a_2 r k_2 & r k_2 \end{pmatrix} \geq \begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix}$$

for a sufficiently small positive number ϵ and sufficiently large $\xi < 0$.

Integrating the above inequality (3.6) from y to ξ (close to $-\infty$) we have

$$\begin{aligned} c \left(\begin{pmatrix} \tilde{u}(\xi) \\ \tilde{v}(\xi) \end{pmatrix} - \begin{pmatrix} \tilde{u}(y) \\ \tilde{v}(y) \end{pmatrix} \right) &\geq \int_y^\xi \left(-J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right) ds \\ &\quad + \begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix} \int_y^\xi \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(s) ds \end{aligned}$$

or equivalently

$$(3.7) \quad \begin{aligned} &\begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix} \int_y^\xi \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(s) ds \\ &\leq c \left(\begin{pmatrix} \tilde{u}(\xi) \\ \tilde{v}(\xi) \end{pmatrix} - \begin{pmatrix} \tilde{u}(y) \\ \tilde{v}(y) \end{pmatrix} \right) + \int_y^\xi \left(J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right) ds \end{aligned}$$

Since \tilde{u}, \tilde{v} are both bounded, the left hand side of the above inequality is bounded as long as the last integral in the right hand side is finite.

By Fubini's Theorem and Lebesgue's Dominated Convergence Theorem

$$\begin{aligned}
\int_y^\xi \left(J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right)(s) ds &= - \int_y^\xi \int_{\mathbb{R}} J(z) \left(\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(s) - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(s-z) \right) dz ds \\
&= - \int_y^\xi \int_{\mathbb{R}} z J(z) \int_0^1 \begin{pmatrix} \tilde{u}' \\ \tilde{v}' \end{pmatrix}(s-tz) dt dz ds \\
&= - \int_{\mathbb{R}} z J(z) \int_0^1 \begin{pmatrix} \tilde{u}(\xi-tz) - \tilde{u}(y-tz) \\ \tilde{v}(\xi-tz) - \tilde{v}(y-tz) \end{pmatrix} dt dz \\
&\rightarrow - \int_{\mathbb{R}} z J(z) \int_0^1 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(\xi-tz) dt dz
\end{aligned}$$

as $y \rightarrow -\infty$, therefore

$$\int_y^\xi \left(J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right)(s) ds$$

is bounded.

This shows that

$$\begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix} \int_y^\xi \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(s) ds$$

is bounded, which implies that

$$\int_y^\xi \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(s) ds$$

is also bounded. This is due to the fact that the matrix

$$\begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix}^{-1} = \frac{1}{D} \begin{pmatrix} r k_2 - \epsilon & a_2 r k_2 + \epsilon \\ a_1 k_1 + \epsilon & k_1 - \epsilon \end{pmatrix}$$

has all positive entries and $D = (k_1 - \epsilon)(r k_2 - \epsilon) - [-a_1 k_1 - \epsilon][-a_2 r k_2 - \epsilon] > 0$ for sufficiently small $0 < \epsilon < \min \left\{ \frac{r(1-a_1)(1-a_2)}{1-a_1^2+r(1-a_2^2)}, \frac{r(1-a_2)}{1-a_1 a_2} \right\}$. Thus $(\tilde{u}, \tilde{v})^T(\xi)$ is integrable at $-\infty$.

Step 2. We show that the function $(w_1(\xi), w_2(\xi)) =: \int_{-\infty}^\xi (\tilde{u}(s), \tilde{v}(s)) ds$ is integrable for ξ close to $-\infty$. By step 3, $(w_1, w_2)(\xi)$ is well defined and convergent.

Let $\delta = \max_{0 \leq \tilde{u} \leq a_1 k_2, 0 \leq \tilde{v} \leq k_2} \{\tilde{u} + k_1 - a_1 \tilde{v}, r(a_2 \tilde{u} + k_2 - \tilde{v})\} + 1$ and $\beta = \delta/c$.

Firstly we show the function

$$e^{-\beta \xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(\xi)$$

is monotonically decreasing.

Since

$$\begin{aligned}
\left[e^{-\beta\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \right]' &= e^{-\beta\xi} \begin{pmatrix} \tilde{u}' - \beta\tilde{u} \\ \tilde{v}' - \beta\tilde{v} \end{pmatrix} (\xi) \\
&= \frac{1}{c} e^{-\beta\xi} \begin{pmatrix} -J * \tilde{u} + \tilde{u} + (-k_1 - \tilde{u})(a_1\tilde{v} - \tilde{u}) - \delta\tilde{u} \\ -J * \tilde{v} + \tilde{v} + r(k_2 - \tilde{v})(-a_2\tilde{u} + \tilde{v}) - \delta\tilde{v} \end{pmatrix} (\xi) \\
&\leq 0,
\end{aligned}$$

it follows that the function $e^{-\beta\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi)$ is decreasing.

It is easy to see that $J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s)ds$ is also integrable on $(-\infty, \xi)$ for ξ sufficiently negatively large. Integrating (3.7) from $-\infty$ to ξ we get

$$\begin{aligned}
&\begin{pmatrix} k_1 - \epsilon & -a_1k_1 - \epsilon \\ -a_2rk_2 - \epsilon & rk_2 - \epsilon \end{pmatrix} \int_{-\infty}^{\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s)ds \\
&\leq c \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) + \int_{-\infty}^{\xi} \left(J * \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right) ds
\end{aligned}$$

Integrating this inequality again from y to ξ we get

$$\begin{aligned}
&\begin{pmatrix} k_1 - \epsilon & -a_1k_1 - \epsilon \\ -a_2rk_2 - \epsilon & rk_2 - \epsilon \end{pmatrix} \int_y^{\xi} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s) \\
&\leq c \int_y^{\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s)ds + \int_y^{\xi} \left(J * \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) ds
\end{aligned}$$

Since J has compact support, we can again use Fubini's Theorem and Lebesgue's Theorem to obtain

$$\begin{aligned}
&\int_y^{\xi} \int_{\mathbb{R}} J(t) \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s) - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s-t) \right) dt ds \\
&\rightarrow \int_{\mathbb{R}} t J(t) \int_0^1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi - \theta t) d\theta dt
\end{aligned}$$

as $y \rightarrow -\infty$. Thus $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s)$ is integrable on $(-\infty, \xi)$.

Step 3. The function (\tilde{u}, \tilde{v}) decays at least exponentially at $-\infty$.

Similar to step 2, we have

$$\begin{aligned}
& \begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix} \int_{-\infty}^{\xi} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s) \\
\leq & c \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) - \int_{\mathbb{R}} t J(t) \int_0^1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi - \theta t) d\theta dt \\
\leq & c \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \int_{-\infty}^0 |t| J(t) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi - t) dt \\
= & c \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \int_{-\infty}^0 |t| J(t) \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \int_{\xi}^{\xi-t} e^{\beta s} e^{-\beta s} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s) ds \right) dt \\
\leq & \left(c + \int_{-\infty}^0 |t| J(t) dt \right) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \frac{1}{\beta} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \int_{-\infty}^0 |t| J(t) (e^{-\beta t} - 1) dt.
\end{aligned}$$

Here we have used the facts that $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}' (\xi) \geq 0$ and $e^{-\beta \xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi)$ is decreasing.

Furthermore,

$$\begin{aligned}
& \frac{1}{\beta} \int_{-\infty}^{\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s) ds \int_{-\infty}^0 |t| J(t) dt + \begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix} \int_{-\infty}^{\xi} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s) \\
\leq & \frac{1}{\beta} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \int_{-\infty}^0 |t| J(t) dt + \begin{pmatrix} k_1 - \epsilon & -a_1 k_1 - \epsilon \\ -a_2 r k_2 - \epsilon & r k_2 - \epsilon \end{pmatrix} \int_{-\infty}^{\xi} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s) \\
\leq & \left(c + \int_{-\infty}^0 |t| J(t) dt \right) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \frac{1}{\beta} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \int_{-\infty}^0 |t| J(t) e^{-\beta t} dt.
\end{aligned}$$

Hence there exists a positive matrix M (see step 3) with all positive entries such that

$$\int_{-\infty}^{\xi} \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s) \right) ds \leq M \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \right)$$

Noting that $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s)$ and $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s)$ are increasing on $(-\infty, \xi)$, we have

$$\begin{aligned}
M \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \right) & \geq \int_{\xi-r}^{\xi} \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (s) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s) \right) ds \\
& \geq r \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi - r) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi - r) \right)
\end{aligned}$$

Thus for large r , there is some $0 < k < 1$ such that

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi - r) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi - r) \leq k \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \right)$$

Let $H(\xi) = e^{-\gamma\xi} \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \right)$, where $\gamma = \frac{1}{r} \ln \frac{1}{k}$, then

$$\begin{aligned} H(\xi - r) &= e^{-\gamma(\xi - r)} \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi - r) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi - r) \right) \\ &= e^{-\gamma\xi} e^{\gamma r} \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi - r) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi - r) \right) \\ &\leq e^{-\gamma\xi} \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \right) = H(\xi) \end{aligned}$$

The fact that $0 \leq \tilde{u} \leq a_1 k_2$ and $0 \leq \tilde{v} \leq k_2$ implies

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) = \int_{-\infty}^{\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (s) ds \leq \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (0) + \begin{pmatrix} a_1 k_2 \xi \\ k_2 \xi \end{pmatrix} \quad \text{for } \xi > 0,$$

and thus $\lim_{\xi \rightarrow +\infty} H(\xi) = 0$, and $H(\xi)$ is bounded. Consequently we have

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\xi) + \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \leq O(e^{\gamma\xi}) \text{ as } \xi \rightarrow -\infty$$

and therefore

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} (\xi) \leq O(e^{\gamma\xi})$$

for $\xi \rightarrow -\infty$. □

The inequality $(u, v)(\xi) \leq O(e^{\gamma\xi})$ says that the two components of the traveling front may approach the equilibrium with different rates, however, we will show this will not happen and give the exact decay rates of the front solutions. Recalling (see the proof of the previous Lemma) that at $-\infty$ the limit matrix of the Jacobian of the vector functions $((-k_1 - \tilde{u})(a_1 \tilde{v} - \tilde{u}), r(k_2 - \tilde{v})(-a_2 \tilde{u} + \tilde{v}))$ is

$$\bar{M} = \begin{pmatrix} k_1 & -a_1 k_1 \\ -a_2 r k_2 & r k_2 \end{pmatrix},$$

and it has two positive eigenvalues

$$\mu_{1,2} = \frac{k_1 + r k_2 \pm \sqrt{(k_1 + r k_2)^2 - 4 r k_1 k_2 (1 - a_1 a_2)}}{2} > 0.$$

For any $c > 0$ each of the following equations

$$\tilde{M}_{11} = \int_{\mathbb{R}} J(s) e^{\lambda s} ds + c\lambda - 1 - \mu_1 = 0, \quad \tilde{M}_{22} = \int_{\mathbb{R}} J(s) e^{\lambda s} ds + c\lambda - 1 - \mu_2 = 0$$

has exactly one positive root. For $c > c^*$ ($c = c^*$) let $\bar{\mu}(c)$ ($\bar{\mu}(c^*)$) be the smaller positive root of the above equations. We also denote the smaller positive root of the equation

$$M_-(\lambda) = \int_{\mathbb{R}} J(\xi) e^{\lambda \xi} d\xi - c\lambda + r(1 - a_2) - 1$$

by $\lambda_1(c)$ for $c > c^*$ and its double positive root as λ^* for $c = c^*$.

Theorem 12. *For every $c \geq c^*$, the traveling wave solution as derived in Theorem 7 has the following asymptotic properties:*

(i). *Corresponding to the wave speed $c > c^*$,*

$$(3.8) \quad \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\lambda_1(c)\xi} + o(e^{\lambda_1(c)\xi})$$

as $\xi \rightarrow -\infty$; and

$$(3.9) \quad \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} a_1 k_2 \\ k_2 \end{pmatrix} - \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} e^{-\bar{\mu}(c)\xi} + o(e^{-\bar{\mu}(c)\xi})$$

as $\xi \rightarrow +\infty$, where $A_1, A_2, \bar{A}_1, \bar{A}_2$ are positive constants.

(ii). *Corresponding to the wave speed $c = c^*$,*

$$(3.10) \quad \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} A_{11} + A_{12}\xi \\ A_{21} + A_{22}\xi \end{pmatrix} e^{\lambda^*\xi} + o(\xi e^{\lambda^*\xi})$$

as $\xi \rightarrow -\infty$; and

$$(3.11) \quad \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} a_1 k_2 \\ k_2 \end{pmatrix} - \begin{pmatrix} \bar{A}_{11} \\ \bar{A}_{22} \end{pmatrix} e^{-\bar{\mu}(c^*)\xi} + o(e^{-\bar{\mu}(c^*)\xi})$$

as $\xi \rightarrow +\infty$, where $A_{12}, A_{22} < 0$, $A_{11}, A_{21} \in \mathbb{R}$ and $\bar{A}_{11}, \bar{A}_{21} > 0$.

Proof. For every $c \geq c^*$, the traveling front solution $(u(\xi), v(\xi))^T$ to the equation (1.5) satisfies

$$\begin{pmatrix} \underline{u}(\xi) \\ \underline{v}(\xi) \end{pmatrix} \leq \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} \leq \begin{pmatrix} \bar{u}(\xi) \\ \bar{v}(\xi) \end{pmatrix} \quad \xi \in \mathbb{R},$$

where $(\underline{u}, \underline{v})$ and (\bar{u}, \bar{v}) are lower and upper solutions of (1.5).

Lemma 5 implies that the upper- and lower-solutions as derived in Lemma 3 and Lemma 4 have the same asymptotic rate at $\xi \rightarrow -\infty$. Then (3.8) and (3.10) follow.

We next prove (3.9) and (3.11). By Lemma 10, the traveling front approaches the equilibrium $(a_1 k_2, k_2)$ at least exponentially, now we determine the exponential rate. It is convenient to start with the derivative of the traveling wave $(v_1, v_2)(\xi) \doteq (u', v')(\xi)$ which satisfies the system

$$(3.12) \quad \begin{cases} J * v_1 - v_1 - cv'_1 + (2u - a_1 v - 1)v_1 = a_1(u - 1)v_2, \\ J * v_2 - v_2 - cv'_2 + [r(1 - a_2) + ra_2 u - 2rv]v_2 = -a_2 r v v_1 \end{cases}$$

It is easy to see that $v_1(\xi)$ can not decay slower than $v_2(\xi)$. For, if we divide both sides of the second equation of (3.12), and take limit for $\xi \rightarrow -\infty$, we would have a finite number in the left hand side and infinity at the right hand side. A contradiction. Similarly we can show that v_2 can not decay slower than v_1 by the first equation of (3.12). This implies that $a_1 k_2 - u = O(k_2 - v)$ as $\xi \rightarrow +\infty$.

We can change the matrix \bar{M} into diagonal by Jordan canonical transform, in fact let

$$P = \begin{pmatrix} -a_1 k_1 & -a_1 k_1 \\ \mu_1 - k_1 & \mu_2 - k_1 \end{pmatrix}$$

then

$$P^{-1} = \frac{1}{a_1 k_1 (\mu_1 - \mu_2)} \begin{pmatrix} \mu_2 - k_1 & k_1 - \mu_1 \\ a_1 k_1 & -a_1 k_1 \end{pmatrix} \quad \text{so that} \quad P^{-1} \bar{M} P = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Now writing system (3.4) in vector form with $U = (\tilde{u}, \tilde{v})$ and $F = ((-k_1 - \tilde{u})(a_1 \tilde{v} - \tilde{u}), r(k_2 - \tilde{v})(-a_2 \tilde{u} + \tilde{v}))$:

$$J * U - U + cU' - F(U) = 0.$$

Introduce the transformation

$$(3.13) \quad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}(\xi) = P \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(\xi),$$

then $W = (w_1, w_2)$ satisfies the following equation

$$J * W - W + cW' - P^{-1}F(PW) = 0$$

or equivalently

$$(3.14) \quad J * W - W + cW' - \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} W = P^{-1}F(PW) - \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} W.$$

It is easy to verify that as $\xi \rightarrow -\infty$,

$$(3.15) \quad P^{-1}F(PW) - \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} W = O(w_1^2 + w_2^2)$$

Since w_1 and w_2 are linear combinations of \tilde{u} and \tilde{v} we have $(w_1, w_2) \leq O(e^{\gamma\xi})$ as $\xi \rightarrow -\infty$. It then follows that for $0 < Re\lambda < \gamma$ the two side Laplace transforms

$$W_1(\lambda) = \int_{\mathbb{R}} e^{-\lambda\xi} w_1(\xi) d\xi \quad \text{and} \quad W_2(\lambda) = \int_{\mathbb{R}} e^{-\lambda\xi} w_2(\xi) d\xi$$

are well defined. Note that

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\lambda\xi} J * W(\xi) d\xi &= \int_{\mathbb{R}} J(s) e^{\lambda s} \int_{\mathbb{R}} W(\xi + s) e^{-\lambda(\xi+s)} d\xi ds \\
&= \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} (\lambda) \int_{\mathbb{R}} J(s) e^{\lambda s} ds,
\end{aligned}$$

we then have

$$(3.16) \quad \tilde{M}(c, \lambda) \begin{pmatrix} W_1(\lambda) \\ W_2(\lambda) \end{pmatrix} = \int_{\mathbb{R}} e^{-\lambda\xi} (P^{-1}F(PW) - \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} W) d\xi$$

where the matrix \tilde{M} is given by

$$\tilde{M}(c, \lambda) = \begin{pmatrix} \int_{\mathbb{R}} J(s) e^{\lambda s} ds + c\lambda - 1 - \mu_1 & 0 \\ 0 & \int_{\mathbb{R}} J(s) e^{\lambda s} ds + c\lambda - 1 - \mu_2 \end{pmatrix}$$

and from (3.15) we see the right hand side of (3.16) is well defined for $0 < \operatorname{Re}\lambda < 2\gamma$. According to [p 58 Theorem 5a of D. V. Widder], there is a positive $\tilde{\lambda}$ such that $W_1(\lambda)$ and $W_2(\lambda)$ are analytic for $0 < \operatorname{Re}\lambda < \tilde{\lambda}$ and one of W_1 and W_2 has singularity at $\lambda = \tilde{\lambda}$. This shows that for $c \geq c^*$, W_1 and W_2 are defined for $\operatorname{Re}\lambda < \bar{\mu} = \min\{\bar{\mu}_1, \bar{\mu}_2\}$ where $\bar{\mu}_i$, $i = 1, 2$ are the positive root of the two equation:

$$(3.17) \quad \tilde{M}_{11} = \int_{\mathbb{R}} J(s) e^{\lambda s} ds + c\lambda - 1 - \mu_1 = 0,$$

and

$$(3.18) \quad \tilde{M}_{22}(\lambda) = \int_{\mathbb{R}} J(s) e^{\lambda s} ds + c\lambda - 1 - \mu_2 = 0$$

respectively.

Now Let $P^{-1}F(PW) = (\bar{F}_1(w_1, w_2), \bar{F}_2(w_1, w_2))$ and write

$$(3.19) \quad \int_0^{+\infty} w_1(-\xi) e^{\lambda\xi} d\xi = \frac{\int_{\mathbb{R}} e^{-\lambda\xi} [\bar{F}_1(w_1, w_2) - \mu_1 w_1] d\xi}{\tilde{M}_{11}(c, \lambda)} - \int_0^{\infty} w_1(\xi) e^{-\lambda\xi} d\xi,$$

and

$$(3.20) \quad \int_0^{+\infty} w_2(-\xi) e^{\lambda\xi} d\xi = \frac{\int_{\mathbb{R}} e^{-\lambda\xi} [\bar{F}_2(w_1, w_2) - \mu_2 w_2] d\xi}{\tilde{M}_{22}(c, \lambda)} - \int_0^{\infty} w_2(\xi) e^{-\lambda\xi} d\xi.$$

Observing that the second terms $\int_0^\infty w_{1,2}(\xi)e^{-\lambda\xi}d\xi$ on the right hand side of (3.19) and (3.20) are analytic for $Re\lambda > 0$, and there is no positive root of (3.17) ((3.18)) with $Re\lambda = \bar{\mu}_1$ ($Re\lambda = \bar{\mu}_2$) other than $\lambda = \bar{\mu}_1$ ($\lambda = \bar{\mu}_2$, respectively). To see this ([4, 19]), we let $\lambda = \bar{\mu}_1 + i\beta$, $\beta \in \mathbb{R}$ such that

$$\begin{aligned}
0 &= \tilde{M}_{11}(\bar{\mu}_1 + i\beta) \\
&= \int_{\mathbb{R}} e^{(\bar{\mu}_1 + i\beta)y} J(y) dy - c(\bar{\mu}_1 + i\beta) - 1 - \mu_1 \\
&= \int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) \cos \beta y dy - c\bar{\mu}_1 - 1 - \mu_1 + (\int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) \sin \beta y dy - c\beta)i \\
&= \int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) (-2 \sin^2 \frac{\beta y}{2} + 1) dy - c\bar{\mu}_1 - 1 - \mu_1 + (\int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) \sin \beta y dy - c\beta)i \\
&= -2 \int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) \sin^2 \frac{\beta y}{2} dy + (\int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) \sin \beta y dy - c\beta)i
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) \sin^2 \frac{\beta y}{2} dy = 0, \quad \text{and} \quad \int_{\mathbb{R}} e^{\bar{\mu}_1 y} J(y) \sin \beta y dy - c\beta = 0,$$

therefore $\beta = 0$. We can also show that $\tilde{M}_{22}(\lambda)$ has a similar property.

Hence we can apply the modified Ikehara's Tauberian Theorem (see [4]) to the function w_1 and w_2 separately to get

$$w_i = \hat{A}_i e^{\bar{\mu}_i \xi} + o(e^{\bar{\mu}_i \xi}), \quad i = 1, 2$$

for $\hat{A}_i > 0$, $i = 1, 2$.

By (3.13), \tilde{u} and \tilde{v} are linear combinations of w_1 and w_2 . We see that they both decay exponentially at a rate of $\bar{\mu} = \min\{\bar{\mu}_1, \bar{\mu}_2\}$ at $-\infty$. Switching back to u , v from \tilde{u} and \tilde{v} and reversing back the sign of ξ , we have the conclusion of the theorem. \square

We introduce a comparison principle for non-local system that will be used in showing the uniqueness of the traveling front solution of (1.5).

Lemma 13. *Let the C^1 vector function $\bar{U}(\xi) = (\bar{u}_1(\xi), \bar{u}_2(\xi), \dots, \bar{u}_n(\xi))^T$ and $\underline{U}(\xi) = (\underline{u}_1(\xi), \underline{u}_2(\xi), \dots, \underline{u}_n(\xi))^T$ be monotonically increasing in \mathbb{R} and satisfy the following inequalities*

$$D(\bar{U}) - c\bar{U}' + F(\bar{U}) \leq D(\underline{U}) - c\underline{U}' + F(\underline{U}) \quad \text{for } \xi \in [-N, N]$$

and

$$\underline{U}(-N) \leq \bar{U}(\xi), \quad \underline{U}(\xi) \leq \bar{U}(N) \quad \text{for } \xi \in (-N, N]$$

$$\underline{U}(\xi) < \bar{U}(\xi) \quad \text{for } \xi \in (-\infty, -N] \cup [N, +\infty),$$

where $D(U)$ is a diagonal matrix with entries $D_i(J_i * u_i - u_i)$, $D_i > 0$ and J_i is a positive, even integration kernel with unite mass for $i = 1, 2, \dots, n$, $F(U) =$

$(F_1(U), \dots, F_n(U))^T$ is C^1 with respect to its components and $\frac{\partial F_i}{\partial u_j} \geq 0$ for $i \neq j$, $i, j = 1, 2, \dots, n$, then

$$\underline{U}(\xi) \leq \overline{U}(\xi) \quad \text{for } \xi \in [-N, N].$$

Proof. See [16, Lemma 12]. \square

Theorem 14. Assume that $0 < a_1, a_2 < 1$ and (2.7), for every $c \geq c^*$, system (1.5) has corresponding a unique (up to a translation of the origin) traveling wave solution.

Proof. We only prove the conclusion for traveling wave solutions with asymptotic rates (3.8) and (3.9), since the other case can be dealt similarly. Let $U_1(\xi) = (u_1(\xi), v_1(\xi))^T$ and $U_2(\xi) = (u_2(\xi), v_2(\xi))^T$ be two traveling front solutions of system (1.5) with the same speed $c > c^*$. Then there exist positive constants $A_i, B_i, i = 1, 2, 3, 4$ and a large number $N > 0$ such that for $\xi < -N$,

$$(3.21) \quad U_1(\xi) = \begin{pmatrix} A_1 e^{\lambda_1 \xi} \\ A_2 e^{\lambda_1 \xi} \end{pmatrix} + o(e^{\lambda_1 \xi})$$

$$(3.22) \quad U_2(\xi) = \begin{pmatrix} A_3 e^{\lambda_1 \xi} \\ A_4 e^{\lambda_1 \xi} \end{pmatrix} + o(e^{\lambda_1 \xi});$$

and

$$(3.23) \quad U_1(\xi) = \begin{pmatrix} a_1 k_2 - B_1 e^{\lambda_3 \xi} \\ k_2 - B_2 e^{\lambda_3 \xi} \end{pmatrix} + o(e^{\lambda_3 \xi})$$

$$(3.24) \quad U_2(\xi) = \begin{pmatrix} a_1 k_2 - B_3 e^{\lambda_3 \xi} \\ k_2 - B_4 e^{\lambda_3 \xi} \end{pmatrix} + o(e^{\lambda_3 \xi}).$$

The traveling front solutions of system (1.5) are translation invariant, thus for any $\theta > 0$, $U_1^\theta(\xi) := U_1(\xi + \theta)$ is also a traveling wave solution of (1.5). By (3.21) and (3.23), the solution $U_1(\xi + \theta)$ has the asymptotic rates

$$(3.25) \quad U_1^\theta(\xi) = \begin{pmatrix} A_1 e^{\lambda_1 \theta} e^{\lambda_1 \xi} \\ A_2 e^{\lambda_1 \theta} e^{\lambda_1 \xi} \end{pmatrix} + o(e^{\lambda_1 \xi})$$

for $\xi \leq -N$;

$$(3.26) \quad U_1^\theta(\xi) = \begin{pmatrix} a_1 k_2 - B_1 e^{\lambda_3 \theta} e^{\lambda_3 \xi} \\ k_2 - B_2 e^{\lambda_3 \theta} e^{\lambda_3 \xi} \end{pmatrix} + o(e^{\lambda_3 \xi})$$

for $\xi \geq N$.

Choosing $\theta > 0$ large enough such that

$$(3.27) \quad A_1 e^{\lambda_1 \theta} > A_3,$$

$$(3.28) \quad A_2 e^{\lambda_1 \theta} > A_4,$$

$$(3.29) \quad B_1 e^{\lambda_3 \theta} < B_3,$$

$$(3.30) \quad B_2 e^{\lambda_3 \theta} < B_4.$$

then one has

$$(3.31) \quad U_1^\theta(\xi) > U_2(\xi)$$

for $\xi \in (-\infty, -N] \cup [N, +\infty)$.

While on the interval $[-N, N]$ we can apply Lemma 13 to have $U_1^\theta(\xi) > U_2(\xi)$ for $\xi \in [-N, N]$. This shows that

$$U_1^\theta(\xi) > U_2(\xi), \quad \xi \in \mathbb{R}.$$

Now, decrease θ until one of the following situations happens.

(i) There exists a $\bar{\theta} \geq 0$ such that $U_1^{\bar{\theta}}(\xi) \equiv U_2(\xi)$. In this case we have finished the proof.

(ii) There exists a $\bar{\theta} \geq 0$ and $\xi_1 \in \mathbb{R}$ such that one of the components of $U^{\bar{\theta}}$ and U_2 are equal there; and for all $\xi \in \mathbb{R}$, we have $U_1^{\bar{\theta}}(\xi) \geq U_2(\xi)$. One applying the Maximum Principle on \mathbb{R} for that component, we find $U_1^{\bar{\theta}}(\xi)$ and $U_2(\xi)$ must be identical on that component. Without loss of generality, we suppose that the component is the first component. Then $U_1^{\bar{\theta}}(\xi) - U_2(\xi)$ satisfies (3.1) and (3.3). Plugging $w_1 \equiv 0$ into (3.1) again we find that there is at least one $\xi_{\bar{\theta}}$ such that $W_2(\xi_{\bar{\theta}}) = 0$. Then by applying Maximum Principle to (3.2), we have $w_2(\xi) \equiv 0$ for $\xi \in \mathbb{R}$. We have then return to case (i).

Consequently, in either situation, there exists a $\bar{\theta} \geq 0$ such that

$$U_1^{\bar{\theta}}(\xi) \equiv U_2(\xi)$$

for all $\xi \in \mathbb{R}$. □

The next theorem shows that lower bound c^* for the wave speed c is optimal, hence it is the critical minimal wave speed.

Theorem 15. *There is no monotone traveling wave solution of (1.5) for any $0 < c < c^*$.*

Proof. The proof is basically the same as that of [4] so we skip it. □

REFERENCES

- [1] P. Bates and A. Chmaj, An integro-differential model for phase transitions: Stationary solutions in higher space dimensions, J. Statist. Phys., 95 (1999), 1119-1139.
- [2] P. Bates, P. Fife, X. Ren and X. Wang, Travelling waves in a convolution model for phase transitions. Arch. Rat. Mech. Anal., 138 (1997), 105-136.
- [3] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, Bulletin of the Brazilian Mathematical Society Volume 22, Number 1, 1-37, DOI: 10.1007/BF01244896.

- [4] J. Carr and A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations, *Proc. Amer. Math. Soc.*, 132 (2004), 2433-2439.
- [5] X. Chen, Existence, uniqueness and asymptotic stability of travelling waves in non-local evolution equations, *Adv. Differential Equations*, 2 (1997), 125-160.
- [6] C. Cortazar, M. Elgueta and J. Rossi, A non-local diffusion equation whose solutions develop a free boundary, *Annales Henri Poincaré*, 6 (2005), 269-281.
- [7] C. Cortazar, M. Elgueta, J. Rossi and N. Wolanski, Boundary fluxes for non-local diffusion, *J. Differential Equations*, 234 (2007), 360-390.
- [8] B. Ermentrout and J. McLeod, Existence and uniqueness of traveling waves for a neural network, *Proc. Roy. Soc. Edinburgh*, 123 A (1994), 1013-1022.
- [9] J. Coville, "Equations de réaction diffusion non-locale", Ph.D thesis, University of Paris, 2003.
- [10] J. Coville, Maximum principles, sliding techniques and applications to nonlocal equations, *Electron. J. Diff. Eqns.*, 68 (2007), 1-23.
- [11] J. Coville and L. Dupaigne, Propagation speed of travelling fronts in nonlocal reaction diffusion equations, *Nonl. Anal. TMA* 60 (2005), 797-819.
- [12] J. Coville and L. Dupaigne, On a nonlocal equation arising in population dynamics, *Proc. Roy. Soc. Edinburgh*, 137 (2007), 1-29.
- [13] J. Fang and X. Zhao, Existence and uniqueness of traveling waves for non-monotone integral equations with applications, *J. Differential Equations* 248 (2010), 2199-2226, 45M20 (35C07 92D25).
- [14] P. Fife and X. Wang, A convolution model for interfacial motion: the generation and propagation of internal layers in higher space dimensions, *Adv. Differential Equations*, 3 (1998), 85-110.
- [15] X. Hou, On the minimal speed and asymptotics of the wave solutions for the lotka volterra system, *arXiv:1002.2882*.
- [16] X. Hou and Y. Li, Competition exclusion-a case study of the front solutions to the nonlocal Lotka Volterra system, Preprint.
- [17] X. Hou and W. Feng, Traveling waves and their stability in a coupled reaction diffusion system, *Comm. Pure Appl. Anal.*, 10 (2011), 141-160.
- [18] L. Ignat and J. Rossi, A nonlocal convection-diffusion equation, *J. Functional Analysis*, 251 (2007), 399-437.
- [19] G. Lv, Asymptotic behavior of traveling wave fronts and entire solutions for a nonlocal monostable equation, *Nonlinear Analysis*, 72 (2010), 3659-3668.
- [20] A. de Masi, E. Orlandi, E. Presutti, Traveling fronts in nonlocal evolution equations, *Archive for Rational Mechanics and Analysis*, Volume 132, Issue 2 (1995), pp.143-205.
- [21] M. Mei, C. Ou and X. Zhao, Global stability of monostable traveling waves for non-local time-delayed reaction-diffusion equations. *SIAM J. Math. Anal.* 42 (2010), no. 6, 2762-2790, 35K57 (35B35 35C07 35R09 92D25).
- [22] S. Pan, W. Li and G. Lin, Traveling fronts in nonlocal delayed reaction-diffusion systems and applications, *Z. Angew. Math. Phys.*, 60 (2009), 377-392.
- [23] K. Schumacher, Traveling-front solutions for integro-differential equations, I, *Jurnal Fur Die Reine Und Angewandte Mathematik*, 316 (1979), 54-70.
- [24] L. Zhang, Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neural networks, *J. Diff. Eqns.*, 197 (2004), 162-196.
- [25] L. Zhang and B. Li, Traveling wave solutions in an integro-differential competition model, *Discrete and Continuous Dynamical Systems - Series B (DCDS-B)*, Pages: 417 - 428, Volume: 17 , Issue: 1 , January 2012.

Current address: ¹Department of Mathematics and Statistics, University of North Carolina Wilmington, NC 28403, USA; ²School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China. *Supported by the Fundamental Research Funds for the Central Universities of China and by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.